

Neutrino mass anarchy and the Universe

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ABSTRACT: We study the consequence of the neutrino mass anarchy on cosmology, in particular the total mass of neutrinos and baryon asymmetry through leptogenesis. We require independence of measure in each mass matrix elements in addition to the basis independence, which uniquely picks the Gaussian measure. A simple approximate U(1) flavor symmetry makes leptogenesis highly successful. Correlations between the baryon asymmetry and the light-neutrino quantities are investigated. We also discuss possible implications of recently suggested large total mass of neutrinos by the SDSS/BOSS data.

KEYWORDS: Beyond Standard Model, Cosmology of Theories beyond the SM, Neutrino Physics, Statistical Methods

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1 Introduction

Neutrino physics is a unique area in particle physics that has many direct consequences on the evolution history and the current state of Universe. It was one of the first hypotheses for the non-baryonic dark matter. Excluding this possibility relied on rather surprising constraint that the density of neutrinos would exceed that allowed by Fermi degeneracy in the core of dwarf galaxies! [1] Because of the free streaming, massive neutrinos would also suppress the large-scale structure, which is still subject to active research. The explosion mechanism of supernova is tied to properties of neutrinos, and hence the chemical evolution of galaxies depend on neutrinos. The number of neutrinos is relevant to Big-Bang Nucleosynthesis. In addition, neutrinos may well have created the baryon asymmetry of the Universe [2] or create the Universe itself with scalar neutrino playing the role of the inflaton [3, 4].

Many of the consequences of neutrino properties on the Universe rely on the mass of neutrinos. After many decades of searches, neutrino mass was discovered in 1998 in disappearance of atmospheric neutrinos by the Super-Kamiokande collaboration [5]. Subsequently the SNO experiment demonstrated transmutation of solar electron neutrinos to other active neutrino species [6] corroborated by reactor neutrino data from KamLAND [7].

Most recently, the last remaining mixing angle was discovered by the Daya Bay reactor neutrino experiment [8]. Other experiments confirmed this discovery [9, 10].

On the other hand, fermion masses and mixings have been a great puzzle in particle physics ever since the discovery of muon. Through decades of intensive studies, we have discovered the existence of three generations and a bizarre mass spectrum and mixings among them. The underlying mechanism for this pattern is still not clear. But the hierarchical masses and small mixings exhibited by quarks and charged leptons seem to suggest that mass matrices are organized by some yet-unknown symmetry principles. The discovery of neutrino masses and mixings seem to even complicate the puzzle. From the current data [11]

$$\Delta m_{12}^2 = (7.50 \pm 0.20) \times 10^{-5} \text{eV}^2, \quad (1.1)$$

$$|\Delta m_{23}^2| = (2.32_{-0.08}^{+0.12}) \times 10^{-3} \text{eV}^2, \quad (1.2)$$

$$\sin^2 2\theta_{23} > 0.95 \text{ (90\% C.L.)}, \quad (1.3)$$

$$\sin^2 2\theta_{12} = 0.857 \pm 0.024, \quad (1.4)$$

$$\sin^2 2\theta_{13} = 0.095 \pm 0.010, \quad (1.5)$$

the neutrinos also display a small hierarchy $\Delta m_{12}^2 / |\Delta m_{23}^2| \sim \frac{1}{30}$,¹ which is quite mild compared to quarks and charged leptons. In addition, unlike quarks and charged leptons, the neutrinos have both large and small mixing angles. Many attempts were made to explain this picture using ordered, highly structured neutrino mass matrices, especially when θ_{13} was consistent with zero [12–16].

Quite counterintuitively, however, it was pointed out that the pattern of neutrino masses and mixings can also be well accounted for by structureless mass matrices [17]. Mass matrices with independently random entries can naturally produce the small hierarchical mass spectrum and the large mixing angles. This provides us with an alternative point of view: instead of contrived models for the mass matrix, one can simply take the random mass matrix as a low energy effective theory [18]. This anarchy approach is actually more naturally expected, because after all, the three generations possess the exact same gauge quantum numbers. From the viewpoint of anarchy, however, the mass spectrum with large hierarchy and small mixings exhibited by quarks and charged leptons need an explanation. It turns out that introducing an approximate U(1) flavor symmetry [19, 20] can solve this problem well [18]. This combination of random mass matrix and approximate U(1) flavor symmetry has formed a new anarchy-hierarchy approach to fermion masses and mixings [18].

To be consistent with the spirit of anarchy, the measure to generate the random mass matrices has to be basis independent [18]. This requires that the measure over the unitary matrices be Haar measure, which unambiguously determines the distributions of the mixing angles and CP phases. Consistency checks of the predicted probability distributions of neutrino mixing angles against the experimental data were also performed in great

¹Throughout this paper, we use *small hierarchy* for a mass spectrum typically like 1 : 3 : 10, and *large hierarchy* for a mass spectrum typically like 1 : 10² : 10⁴. So in our wording, the neutrinos exhibit a small hierarchy, while the quarks and leptons exhibit a large hierarchy

detail [21, 22]. Although quite successful already, this anarchy-hierarchy approach obviously has one missing brick: a choice of measure to generate the eigenvalues of the random mass matrices. With the only restriction being basis independence, one can still choose any measure for the diagonal matrices at will. However, as we will show in this paper, if in addition to basis independence, one also wants the matrix elements to be distributed independent from each other, then the only choice is the Gaussian measure.

In this paper, we focus on the Gaussian measure to investigate the consequences. As pointed out in [18], the quantities most sensitive to this choice would be those closely related to neutrino masses. We study a few such quantities of general interest, including the effective mass of neutrinoless double beta decay m_{eff} , the neutrino total mass m_{total} , and the baryon asymmetry η_{B0} obtained through a standard leptogenesis [23, 24]. We also present a correlation analysis between η_{B0} and light-neutrino parameters. Recently, the correlation between leptogenesis and light-neutrino quantities was also studied by taking linear measure in [25]. Their results are in broad qualitative agreement with ours in this paper.

The rest of this paper is organized as following. We first motivate our sampling model — Gaussian measure combined with approximate U(1) flavor symmetry — in section 2. Then the consequences of this sampling model is presented in section 3. We show our Monte Carlo predictions on light-neutrino mass hierarchy, effective mass of neutrinoless double beta decay, light-neutrino total mass, and baryon asymmetry through leptogenesis. In section 4, we investigate the correlations between baryon asymmetry and light-neutrino quantities. A recent Baryon Oscillation Spectroscopic Survey (BOSS) analysis suggests that the total neutrino mass could be quite large [26]. Our section 5 is devoted to discuss the consequence of this suggestion. We conclude in section 6.

2 Sampling model

2.1 Gaussian measure

To accommodate the neutrino masses, we consider the standard model with an addition of three generations of right-handed neutrinos ν_R , which are singlets under $\text{SU}(2)_L \times \text{U}(1)_Y$ gauge transformations. Then there are two neutrino mass matrices, the *Majorana mass matrix* m_R and the *Dirac mass matrix* m_D , allowed by gauge invariance

$$\begin{aligned} \Delta\mathcal{L} \supset & -\epsilon^{ab}\bar{L}_a H_b^\dagger y_\nu \nu_R - \frac{1}{2}\bar{\nu}_R^c m_R \nu_R + h.c. \\ & \supset -\bar{\nu}_L m_D \nu_R - \frac{1}{2}\bar{\nu}_R^c m_R \nu_R + h.c., \end{aligned} \quad (2.1)$$

where $y_\nu = \frac{\sqrt{2}}{v}m_D$, with $v = 246 \text{ GeV}$. With the spirit of anarchy, we should not discard either of them without any special reason. Both should be considered as random inputs. We parameterize them as

$$m_R = \mathcal{M} \cdot U_R D_R U_R^T, \quad (2.2)$$

$$m_D = \mathcal{D} \cdot U_1 D_0 U_2^\dagger, \quad (2.3)$$

where D_R and D_0 are real diagonal matrices with non-negative elements; U_R , U_1 and U_2 are unitary matrices; \mathcal{M} and \mathcal{D} are overall scales.

At this point, the requirement of neutrino basis independence turns out to be very powerful. It requires that the whole measure of the mass matrix factorizes into that of the unitary matrices and diagonal matrices [18]:

$$dm_R \sim dU_R dD_R, \quad (2.4)$$

$$dm_D \sim dU_1 dU_2 dD_0. \quad (2.5)$$

Furthermore, it also demands the measure of U_R , U_1 and U_2 to be the Haar measure of $U(3)$ group [18]:

$$U = e^{i\eta} e^{i\phi_1 \lambda_3 + i\phi_2 \lambda_8} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \times \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{i\chi_1 \lambda_3 + i\chi_2 \lambda_8}, \quad (2.6)$$

$$dU = ds_{23}^2 dc_{13}^4 ds_{12}^2 d\delta \cdot d\chi_1 d\chi_2 \cdot d\eta d\phi_1 d\phi_2, \quad (2.7)$$

where $\lambda_3 = \text{diag}(1, -1, 0)$, $\lambda_8 = \text{diag}(1, 1, -2)/\sqrt{3}$, and $c_{12} = \cos \theta_{12}$, etc.

Although basis independence is very constraining, it does not uniquely fix the measure choice of m_R or m_D , because arbitrary measure of the diagonal matrices D_R and D_0 is still allowed. Now let us look at the entries of m_R and m_D . There are 9 complex free entries for m_D and 6 complex free entries for the symmetric matrix $m_R = m_R^T$. Generating each free entry independently is probably the most intuitive way of getting a random matrix. However, if one combines this free entry independence with the basis independence requirement, then it turns out there is only one option left — the Gaussian measure:

$$dm_D \sim \prod_{ij} e^{-|m_{D,ij}|^2} dm_{D,ij} = \left(\prod_{ij} dm_{D,ij} \right) e^{-\text{tr}(m_D m_D^\dagger)}, \quad (2.8)$$

$$dm_R \sim \prod_i e^{-|m_{R,ii}|^2} dm_{R,ii} \prod_{i < j} e^{-2|m_{R,ij}|^2} dm_{R,ij} = \left(\prod_{i \leq j} dm_{R,ij} \right) e^{-\text{tr}(m_R m_R^\dagger)}. \quad (2.9)$$

Although this is a known result in random matrix theory [27, 28], we also present our own proof in the appendix.

On one hand, basis independence is necessary from the spirit of anarchy. On the other hand, free entry independence is also intuitively preferred. With these two conditions combined, we are led uniquely to the Gaussian measure. Now the only parameters left free to choose are the two units \mathcal{M} and \mathcal{D} . Following the spirit of anarchy, \mathcal{D} should be chosen to make the neutrino Yukawa coupling of order unity,

$$y_\nu = \frac{\sqrt{2}}{v} m_D \sim O(1). \quad (2.10)$$

Throughout this paper, we choose $\mathcal{D} = 30 \text{ GeV}$, which yields $y_\nu \sim 0.6$. Then we choose \mathcal{M} to fix the next largest mass-square difference of light-neutrinos Δm_l^2 at $2.5 \times 10^{-3} \text{ eV}^2$ in accordance with the data.

right-handed neutrino	U(1) flavor charge
$\nu_{R,1}$	2
$\nu_{R,2}$	1
$\nu_{R,3}$	0

Table 1. The U(1) flavor charge assignments for right-handed neutrinos used in this paper.

2.2 Approximate U(1) flavor symmetry

Our model (eq. (2.1)) is capable of generating a baryon asymmetry η_{B0} through thermal leptogenesis [2, 29]. For the simplicity of analysis, we would like to focus on the scenario with two conditions:

- (1) There is a hierarchy among heavy-neutrino masses $M_1 \ll M_2, M_3$, so that the dominant contribution to η_{B0} is given by the decay of the lightest heavy neutrino N_1 [23].
- (2) If we use h_{ij} to denote the Yukawa couplings of heavy-neutrino mass eigenstates

$$\Delta\mathcal{L} \supset -h_{ij}\epsilon^{ab}\bar{L}_{ai}H_b^\dagger N_j, \quad (2.11)$$

then the condition $h_{i1} \ll 1$ for all $i = 1, 2, 3$ would justify the neglect of annihilation process $N_1 N_1 \rightarrow \bar{l}l$, and also help driving the decay of N_1 out of equilibrium [23, 29]. This condition used to be taken as an assumption [29].

Both conditions above can be achieved by imposing a simple U(1) flavor charge on the right-handed neutrinos. For example, one can make charge assignments as shown in table 1. Assuming a scalar field ϕ carries -1 of this U(1) flavor charge, one can construct neutral combinations ν_ϕ as

$$\nu_\phi = \begin{pmatrix} \nu_{\phi,1} \\ \nu_{\phi,2} \\ \nu_{\phi,3} \end{pmatrix} = \begin{pmatrix} \nu_{R,1} \cdot \phi^2 \\ \nu_{R,2} \cdot \phi \\ \nu_{R,3} \cdot 1 \end{pmatrix}. \quad (2.12)$$

Now it only makes sense to place the random coupling matrices among these neutral combinations

$$\Delta\mathcal{L} \supset -\bar{\nu}_L m_{D0} \nu_\phi - \frac{1}{2} \bar{\nu}_\phi^c m_{R0} \nu_\phi + h.c., \quad (2.13)$$

where m_{R0} and m_{D0} should be generated according to Gaussian measure as in eq. (2.8)–(2.9). Upon U(1) flavor symmetry breaking $\langle \phi \rangle = \epsilon$ with $\epsilon \simeq 0.1$, this gives

$$\nu_\phi \supset \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \nu_R \equiv D_\epsilon \cdot \nu_R, \quad (2.14)$$

and hence the mass matrices

$$m_R = D_\epsilon m_{R0} D_\epsilon = \mathcal{M} \cdot D_\epsilon U_R D_R U_R^T D_\epsilon, \quad (2.15)$$

$$m_D = m_{D0} D_\epsilon = \mathcal{D} \cdot U_1 D_0 U_2^\dagger D_\epsilon. \quad (2.16)$$

Let us parameterize the heavy-neutrino mass matrix as $m_N = U_N M U_N^T$, then

$$m_N \approx m_R \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} \begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}, \quad (2.17)$$

where one can identify

$$M \sim \begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_N \sim \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}. \quad (2.18)$$

Clearly a hierarchy among heavy neutrino masses is achieved. Furthermore, the heavy neutrino mass eigenstates are $N = U_N^T \nu_R$. Since

$$\Delta \mathcal{L} \supset -\epsilon^{ab} \bar{L}_a H_b^\dagger y_\nu \nu_R \equiv -h_{ij} \epsilon^{ab} \bar{L}_{ai} H_b^\dagger N_j, \quad (2.19)$$

we obtain the Yukawa coupling h_{ij} as

$$\begin{aligned} h &= y_\nu U_N^* = \frac{\sqrt{2}}{v} m_D U_N^* \\ &\sim \frac{\sqrt{2}}{v} m_{D0} \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} \\ &\sim \frac{\sqrt{2}}{v} m_{D0} \begin{pmatrix} \epsilon^2 & \epsilon^3 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^2 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}. \end{aligned} \quad (2.20)$$

We see that $h_{i1} \sim \epsilon^2 \ll 1$ is guaranteed for all $i = 1, 2, 3$.

It is worth noting that the light-neutrino mass matrix m_ν is not affected by our U(1) flavor change assignment on right-handed neutrinos (table 1). The hierarchical matrix D_ϵ cancels due to the seesaw mechanism [30–35]:

$$m_\nu = m_D m_R^{-1} m_D^T = m_{D0} m_{R0}^{-1} m_{D0}^T. \quad (2.21)$$

Therefore all properties of light neutrinos do not depend on the particular U(1) flavor charge assignment nor the size of the breaking parameter ϵ . The leptogenesis aspect is the only discussion in the paper where this flavor structure is relevant.

3 Consequences

In this section, we present our Monte Carlo results based on the sampling measure described in the last section.

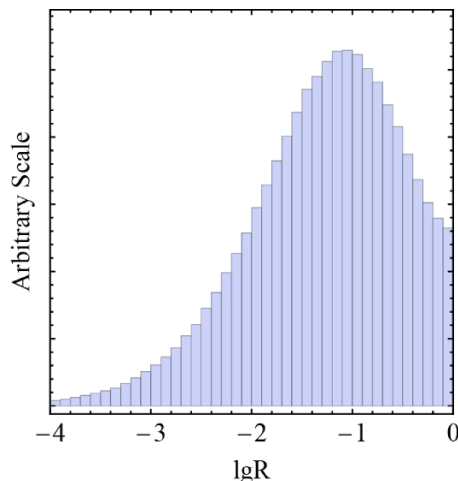


Figure 1. Histogram of $\lg R = \log_{10} R$ with 10^6 occurrences collected.

3.1 Light-neutrino mixings and mass splitting ratio

We parameterize the light neutrino mass matrix as

$$m_\nu = m_D m_R^{-1} m_D^T \equiv U_\nu m U_\nu^T, \quad (3.1)$$

with U_ν a unitary matrix and m a real diagonal matrix with non-negative elements. We also follow a conventional way to label the three masses of the light neutrinos: first sort them in the ascending order $m_{11} \leq m_{22} \leq m_{33}$. Then there are two mass-squared splittings $m_{22}^2 - m_{11}^2$ and $m_{33}^2 - m_{22}^2$. We use Δm_s^2 and Δm_l^2 to denote the smaller and larger one respectively. If Δm_s^2 is the first one, we call this scenario “normal” and take the definitions $m_1 \equiv m_{11}, m_2 \equiv m_{22}, m_3 \equiv m_{33}$. Otherwise, we call it “inverted” and take $m_1 \equiv m_{22}, m_2 \equiv m_{33}, m_3 \equiv m_{11}$. The columns of the unitary matrix U_ν should be arranged accordingly.

Predictions on light-neutrino mixings — the distributions of mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$, CP phase δ_{CP} , and other physical phases χ_1, χ_2 — are certainly the same as in general study of basis independent measures [18], since U_ν is totally governed by the Haar measure. A statistical Kolmogorov-Smirnov test shows that the predicted mixing angle distribution is completely consistent with the experimental data [21, 22].

The mass-squared splitting ratio $R \equiv \Delta m_s^2 / \Delta m_l^2$ is observed to be around [11]

$$R_{\text{exp}} \equiv \frac{7.50 \times 10^{-5}}{2.32 \times 10^{-3}}. \quad (3.2)$$

Figure 1 shows our predicted distribution of this ratio. With a probability of $R < R_{\text{exp}}$ being 36.1%, the prediction is completely consistent with the data.

For the purpose of studying other quantities, we introduce the following *Mixing-Split cuts* as suggested by the experimental data [11] on light-neutrino mixings and mass-squared

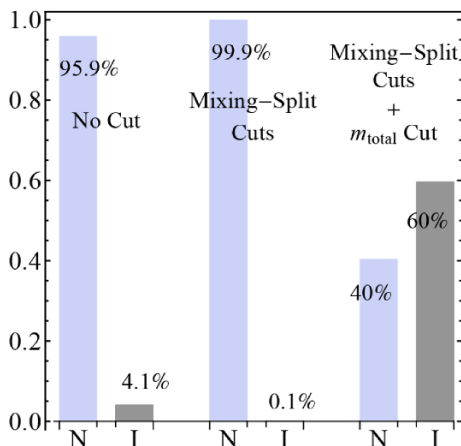


Figure 2. Fractions of normal and inverted mass hierarchy scenarios under different cuts selections, where “N” stands for normal hierarchy and “I” stands for inverted hierarchy. Each of the first two columns consists of 10^6 occurrences, while the last column “Mixing-Split Cuts + m_{total} Cut” contains only 10^4 occurrences.

splitting ratio:

$$\sin^2 2\theta_{23} = 1.0 \quad (3.3)$$

$$\sin^2 2\theta_{12} = 0.857 \quad (3.4)$$

$$\sin^2 2\theta_{13} = 0.095 \quad (3.5)$$

$$R \in R_{\text{exp}} \times (1 - 0.05, 1 + 0.05) \quad (3.6)$$

3.2 Mass hierarchy, m_{eff} and m_{total}

For the mass hierarchy of light-neutrino, our anarchy model predicts normal scenario being dominant. A 10^6 sample Monte Carlo finds the fraction of normal scenario being 95.9% before the Mixing-Split cuts (eq. (3.3)–(3.6)), and 99.9% after applying the cuts. Figure 2 shows the fractions of each scenario.

Neutrino anarchy allows for a random, nonzero Majorana mass matrix m_R . This means that the light neutrinos are Majorana and thus there can be neutrinoless double beta decay process. The effective mass of this process $m_{\text{eff}} \equiv \left| \sum_i m_i U_{\nu,ei}^2 \right|$ is definitely a very broadly interested quantity. Another quantity of general interest is the light-neutrino total mass $m_{\text{total}} \equiv m_1 + m_2 + m_3$, due to its presence in cosmological processes. Our predictions on m_{eff} and m_{total} are shown in figure 3 and figure 4 respectively. For each quantity, we plot both its whole distribution under Gaussian measure and its distribution after applying the Mixing-Split cuts. Distributions of m_{eff} and m_{total} under Gaussian measure were also studied recently in [36]. Their results are in agreement with ours. The small difference is due to the difference in taking cuts on neutrino mixing angles.

We see from figure 3 that most of the time $m_{\text{eff}} \lesssim 0.05$ eV. It becomes even smaller after we apply the Mixing-Split Cuts, mainly below $m_{\text{eff}} \lesssim 0.01$ eV. This is very challenging to experimental sensitivity. For m_{total} , figure 4 shows it being predicted to be very close

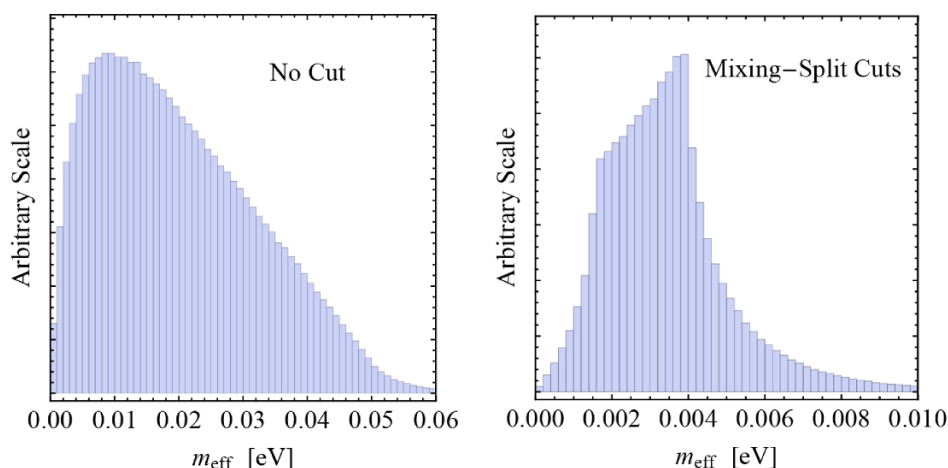


Figure 3. Histogram of m_{eff} with 10^6 occurrences collected. Left/Right panel shows distribution before/after applying the Mixing-Split cuts.

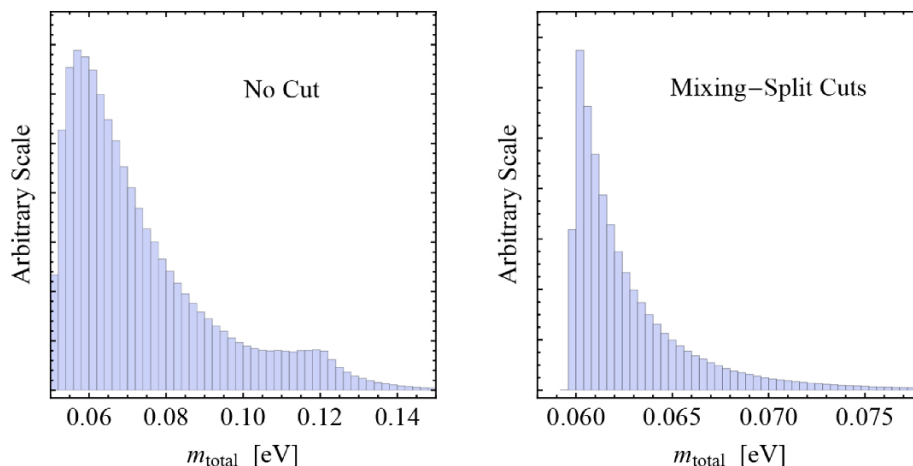


Figure 4. Histogram of m_{total} , with 10^6 occurrences collected. Left/Right panel shows distribution before/after applying the Mixing-Split cuts.

to the current lower bound ~ 0.06 eV. The kink near 0.1 eV is due to the superposition of the two mass hierarchy scenarios.

To understand each component better, we show the distributions of m_{eff} and m_{total} in figure 5 and figure 6 for both before/after applying the cuts and normal/inverted hierarchy scenario. As figure 6 shows, under either hierarchy scenario, m_{total} is likely to reside very close to its corresponding lower bound, especially after applying the cuts. Very interestingly, however, recent BOSS analysis suggests m_{total} could be quite large, with a center value ~ 0.36 eV [26]. As seen clearly from figure 6, a large value of m_{total} would favor inverted scenario. We will discuss some possible consequences of this suggestion in section 5.

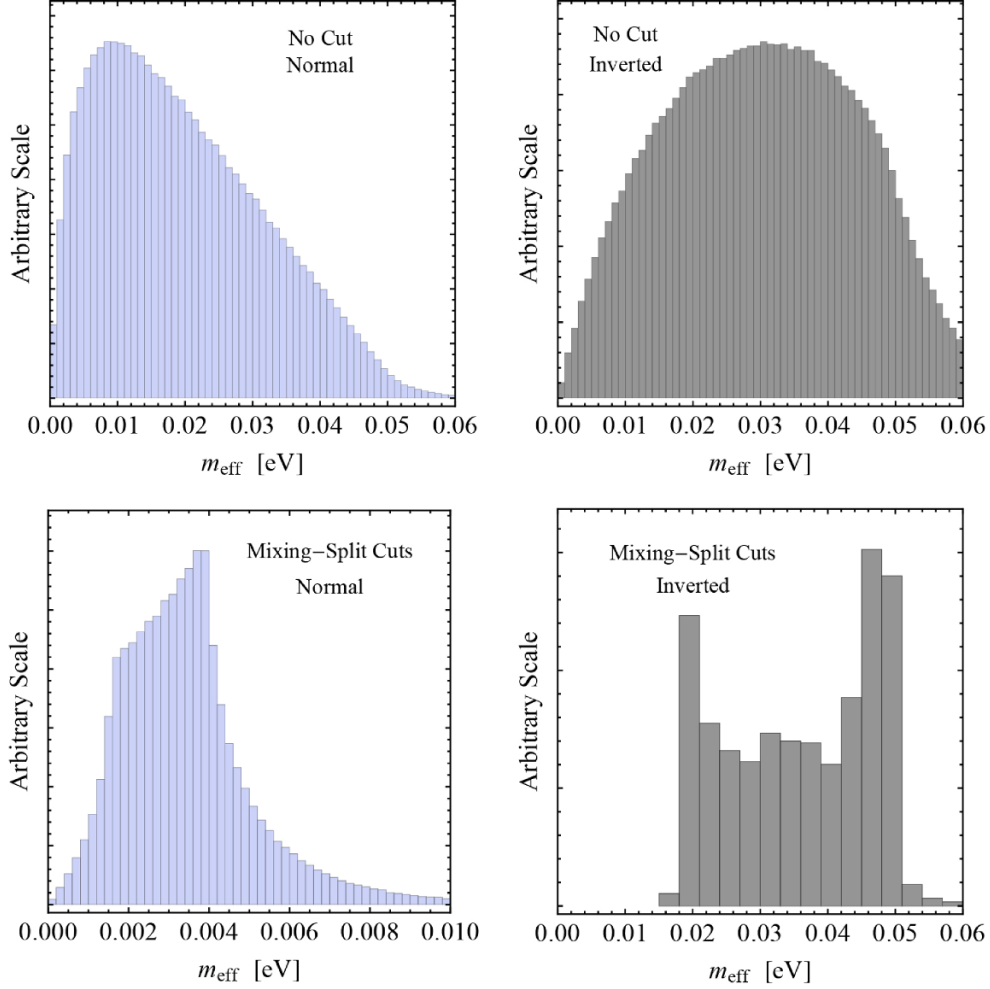


Figure 5. Histogram of m_{eff} with two mass hierarchy scenarios plotted separately. Left/Right column shows distribution under normal/inverted scenario. Upper/Lower row shows distribution before/after applying the Mixing-Split cuts. The plot of inverted scenario with Mixing-Split cuts applied (right bottom) contains 10^4 occurrences, while other plots contain 10^6 occurrences.

3.3 Leptogenesis

As explained in section 2, with our approximate U(1) flavor symmetry, the baryon asymmetry $\eta_{B0} \equiv \frac{n_B}{n_\gamma}$ can be computed through the standard leptogenesis calculations [23, 24]. For each event of m_R and m_D generated, we solve the following Boltzmann equations numerically.

$$\frac{dN_1}{dz} = -(N_1 - N_1^{\text{eq}})(D + S), \quad (3.7)$$

$$\frac{dN_{B-L}}{dz} = -(N_1 - N_1^{\text{eq}})\varepsilon D - N_{B-L}W, \quad (3.8)$$

where we have followed the notations in [23] and [24].

The argument $z \equiv M_1/T$ is evolved from $z_i = 0.001$ to $z_f = 20.0$. Evolving z further beyond 20.0 is not necessary, because the value of N_{B-L} becomes frozen shortly after

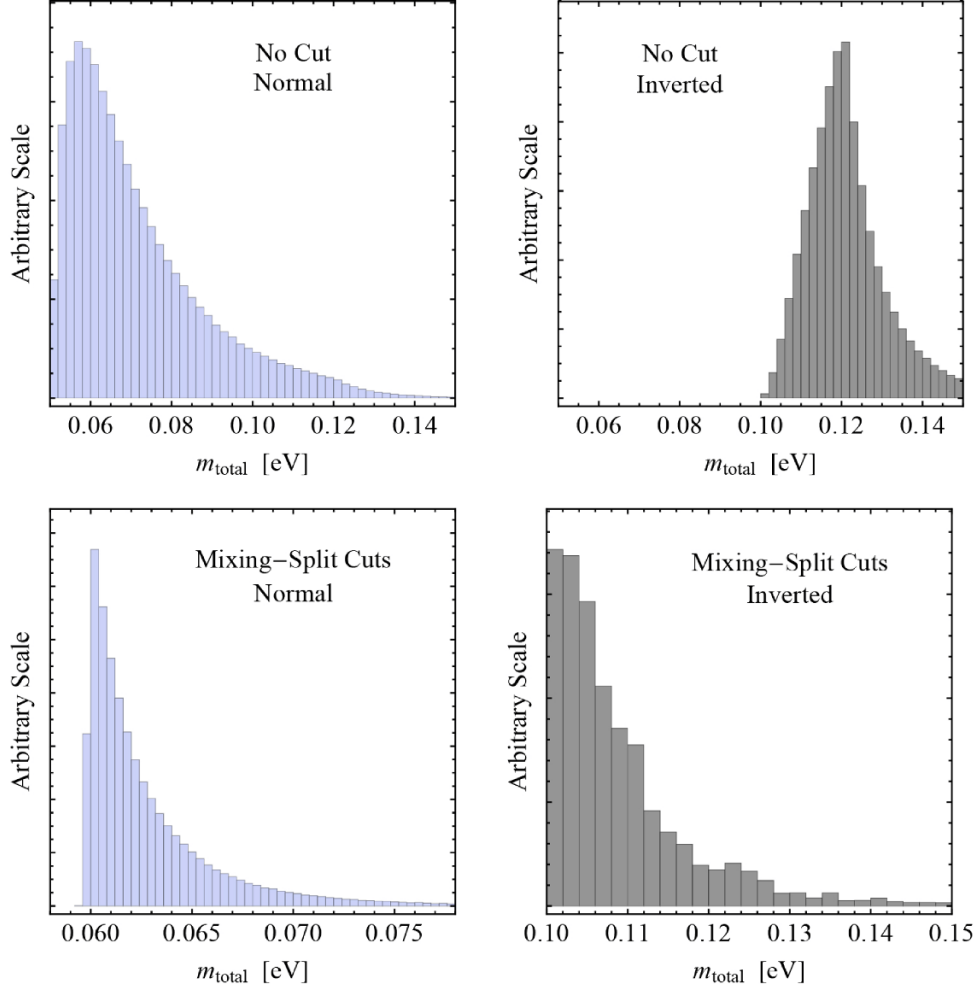


Figure 6. Histogram of m_{total} with two mass hierarchy scenarios plotted separately. Left/Right column shows distribution under normal/inverted scenario. Upper/Lower row shows distribution before/after applying the Mixing-Split cuts. The plot of inverted scenario with Mixing-Split cuts applied (right bottom) contains 10^4 occurrences, while other plots contain 10^6 occurrences.

$z > 10.0$. The baryon asymmetry is then given by $\eta_{B0} = 0.013N_{B-L}^0 \approx 0.013N_{B-L}(z_f)$ [23]. Due to randomly generated m_R and m_D , we have equal chances for ε to be positive or negative. It is the absolute value that is meaningful. We take the initial condition $N_{B-L}(z_i) = 0$. We actually tried two typical initial conditions for N_1 , $N_1(z_i) = 0$ and $N_1(z_i) = N_1^{\text{eq}}(z_i)$, and found no recognizable differences. The distributions of η_{B0} , both before and after applying the Mixing-Split cuts, are shown in figure 7. We see from figure that our prediction on η_{B0} is about the correct order of magnitude as $\eta_{B0,\text{exp}} \approx 6 \times 10^{-10}$ [37].

Let us try to understand the results from some crude estimates. First, let us see why there is almost no difference between the two initial conditions $N_1(z_i) = 0$ and $N_1(z_i) = N_1^{\text{eq}}(z_i)$. The decay function $D(z)$ has the form [24]:

$$D(z) = \alpha_D \frac{K_1(z)}{K_2(z)} z, \quad (3.9)$$

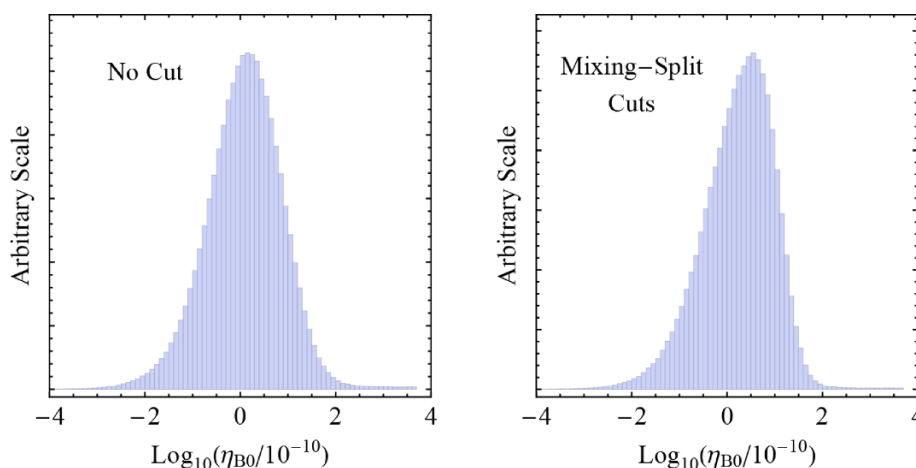


Figure 7. Histogram of η_{B0} with 10^6 occurrences collected. Left/Right panel shows distribution before/after applying Mixing-Split cuts.

where $K_1(z)$ and $K_2(z)$ are modified Bessel functions, and the constant α_D is proportional to the *effective neutrino mass* \tilde{m}_1 :

$$\alpha_D \propto \tilde{m}_1 \equiv \frac{(m_D^\dagger m_D)_{11}}{M_1} . \quad (3.10)$$

So roughly speaking, after $z \gtrsim 1$, the modified Bessel functions become rather close and $D(z)$ increases linearly with z . But in our prescription, the value of $D(z) + S(z)$ becomes already quite large, typically around 50, at $z = 1.0$. So N_1 is forced to be very close to N_1^{eq} thereafter and the solution to the first differential equation (eq. (3.7)) is approximately

$$N_1 - N_1^{\text{eq}} = -\frac{1}{D+S} \frac{dN_1}{dz} \approx -\frac{1}{D+S} \frac{dN_1^{\text{eq}}}{dz} . \quad (3.11)$$

Of course, this initial-condition-independent solution only holds when $D+S$ is large enough, typically for $z > 1.0$. The values of $N_1 - N_1^{\text{eq}}$ at $z < 1.0$ certainly have a considerable dependence on $N_1(z_i)$. However, the solution to the second differential equation (eq. (3.8)) is

$$N_{B-L}^0 = \varepsilon \cdot \int_0^\infty dz \frac{D}{D+S} \frac{dN_1}{dz} e^{-\int_z^\infty W(z') dz'} . \quad (3.12)$$

And due to the shape of $W(z)$, yield of N_{B-L}^0 at $z < 1.0$ is significantly suppressed by a factor $e^{-\int_z^\infty W(z') dz'}$. Therefore η_{B0} turns out to be almost independent of $N_1(z_i)$.

Second, let us crudely estimate the order of magnitude of η_{B0} . In addition to $D(z)$, the scattering functions $S(z)$ and washout function $W(z)$ are also proportional to \tilde{m}_1 . So \tilde{m}_1 is the key factor that significantly affects the evolution of eq. (3.7) and (3.8) [24]. In our anarchy model, apart from the overall units, the mass matrix entries are $O(1)$ numbers, so we expect

$$\tilde{m}_1 = O(1) \cdot \frac{\mathcal{D}^2}{\mathcal{M}} . \quad (3.13)$$

This is just the light-neutrino mass scale. Because in our simulation, \mathcal{M} is chosen such that $\Delta m_l^2 = 2.5 \times 10^3 \text{ eV}^2$, we have

$$\frac{\mathcal{D}^2}{\mathcal{M}} = O(1) \cdot \sqrt{\Delta m_l^2} = O(1) \cdot 0.05 \text{ eV}. \quad (3.14)$$

Therefore most of the time, our model is in the “strong washout regime” [24], since \tilde{m}_1 is much larger than the *equilibrium neutrino mass* m_* :

$$\tilde{m}_1 \sim 0.05 \text{ eV} \gg m_* \sim 10^{-3} \text{ eV}. \quad (3.15)$$

In this regime, the integral in eq. (3.12), which is called *efficiency factor* κ_f , should be around [24]

$$\kappa_f = \int_0^\infty dz \frac{D}{D+S} \frac{dN_1}{dz} e^{-\int_z^\infty W(z') dz'} \sim 10^{-2}. \quad (3.16)$$

Thus our baryon asymmetry is

$$\eta_{B0} = 0.013 N_{B-L}^0 \sim 1.3 \times 10^{-4} \cdot \varepsilon. \quad (3.17)$$

To estimate the CP asymmetry ε , we notice that (following the notation of [24])

$$K \equiv h^\dagger h \sim \left(\frac{\sqrt{2}}{v} \mathcal{D} \right)^2 \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}, \quad (3.18)$$

and thus

$$\begin{aligned} \varepsilon &= \varepsilon_V + \varepsilon_M \sim \frac{3}{16\pi} \sum_{k=1}^3 \frac{\text{Im}(K_{1k}^2)}{K_{11}} \frac{M_1}{M_k} \\ &= \frac{3}{16\pi} \left[\frac{\text{Im}(K_{12}^2)}{K_{11}} \frac{M_1}{M_2} + \frac{\text{Im}(K_{13}^2)}{K_{11}} \frac{M_1}{M_3} \right] \\ &\sim \frac{3}{16\pi} \left(\frac{\sqrt{2}}{v} \mathcal{D} \right)^2 \left(\frac{\epsilon^6}{\epsilon^4} \epsilon^2 + \frac{\epsilon^4}{\epsilon^4} \epsilon^4 \right) \\ &\sim \frac{3}{4\pi} \left(\frac{\mathcal{D}}{v} \right)^2 \epsilon^4 \sim 3 \times 10^{-7}. \end{aligned} \quad (3.19)$$

So the baryon asymmetry is expected to be around $\eta_{B0} \sim 1.3 \times 10^{-4} \cdot \varepsilon \sim 4 \times 10^{-11}$, multiplied by some $O(1)$ factor. This is what we see from figure 7.

Our use of U(1) flavor symmetry breaking plays an essential role to produce the correct order of ε ($\varepsilon \sim \epsilon^4$) and thus η_{B0} . It is thus interesting to investigate what would happen if we had a different U(1) charge assignment. An arbitrary charge assignment would be, upon symmetry breaking, equivalent to an arbitrary choice of D_ϵ parameterized as

$$D_\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} = \epsilon_3 \begin{pmatrix} \epsilon_{r1}\epsilon_{r2} & 0 & 0 \\ 0 & \epsilon_{r2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.20)$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \lesssim 1$, and we have defined $\epsilon_{r2} \equiv \epsilon_2/\epsilon_3$ and $\epsilon_{r1} \equiv \epsilon_1/\epsilon_2$ for convenience. To make the simplest scenario of leptogenesis work, we need the hierarchy among the heavy-neutrino masses. So we restrict ourselves to the case $\epsilon_{r1}, \epsilon_{r2} \ll 1$.

The Majorana mass matrix now becomes

$$m_R = D_\epsilon m_{R0} D_\epsilon \sim \epsilon_3^2 \begin{pmatrix} \epsilon_{r1}^2 \epsilon_{r2}^2 & \epsilon_{r1} \epsilon_{r2}^2 & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2}^2 & \epsilon_{r2}^2 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix} \\ \sim \epsilon_3^2 \begin{pmatrix} 1 & \epsilon_{r1} & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} & 1 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix} \times \begin{pmatrix} \epsilon_{r1}^2 \epsilon_{r2}^2 & 0 & 0 \\ 0 & \epsilon_{r2}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon_{r1} & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} & 1 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix}, \quad (3.21)$$

which gives

$$M \sim \epsilon_3^2 \begin{pmatrix} \epsilon_{r1}^2 \epsilon_{r2}^2 & 0 & 0 \\ 0 & \epsilon_{r2}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.22)$$

$$U_N \sim \begin{pmatrix} 1 & \epsilon_{r1} & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} & 1 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix}. \quad (3.23)$$

The Dirac mass matrix becomes

$$m_D = m_{D0} D_\epsilon, \quad (3.24)$$

which gives the Yukawa coupling h and $K \equiv h^\dagger h$ as

$$h = \frac{\sqrt{2}}{v} m_D U_N^* \\ \sim \frac{\sqrt{2}}{v} m_{D0} \epsilon_3 \begin{pmatrix} \epsilon_{r1} \epsilon_{r2} & \epsilon_{r1}^2 \epsilon_{r2} & \epsilon_{r1}^2 \epsilon_{r2}^2 \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & \epsilon_{r2}^2 \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix} \quad (3.25)$$

$$K \sim \left(\frac{\sqrt{2}}{v} \mathcal{D} \right)^2 \epsilon_3^2 \begin{pmatrix} \epsilon_{r1}^2 \epsilon_{r2}^2 & \epsilon_{r1} \epsilon_{r2}^2 & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2}^2 & \epsilon_{r2}^2 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix}. \quad (3.26)$$

So our ε is given by

$$\varepsilon = \varepsilon_V + \varepsilon_M \sim \frac{3}{16\pi} \sum_{k=1}^3 \frac{\text{Im}(K_{1k}^2)}{K_{11}} \frac{M_1}{M_k} \\ \sim \frac{3}{8\pi} \left(\frac{\sqrt{2}}{v} \mathcal{D} \right)^2 \epsilon_3^2 \epsilon_{r1}^2 \epsilon_{r2}^2 \\ \sim \frac{3}{4\pi} \left(\frac{\mathcal{D}}{v} \right)^2 \epsilon_1^2. \quad (3.27)$$

We see that under the condition $\epsilon_{r1}, \epsilon_{r2} \ll 1$, ε is only sensitive to the value of ϵ_1 .

On the other hand, the value of \tilde{m}_1 is not affected by changing U(1) flavor charge assignments:

$$\begin{aligned}\tilde{m}_1 &\equiv \frac{(m_D^\dagger m_D)_{11}}{M_1} \sim \frac{(m_{D0}^\dagger m_{D0})_{11}}{(M_1)_0} \frac{\epsilon_1^2}{\epsilon_1^2} \\ &= \frac{(m_{D0}^\dagger m_{D0})_{11}}{(M_1)_0} = (\tilde{m}_1)_0.\end{aligned}\tag{3.28}$$

Here a subscript “0” is used to denote the value when there is no U(1) flavor charge assignment, as we did in eq. (2.13). So the strong washout condition (eq. (3.15)) still holds, and we are again led to eq. (3.17). Therefore, the baryon asymmetry η_{B0} can only be affected through ε , which in turn is only sensitive to ϵ_1 , under the condition $\epsilon_{r1}, \epsilon_{r2} \ll 1$.

4 Correlations between η_{B0} and light-neutrino parameters

As we can see from figure 7, the baryon asymmetry is slightly enhanced after applying the Mixing-Split cuts eq. (3.3)–(3.6). This indicates some correlation between η_{B0} and light-neutrino parameters. To understand this better, we would like to systematically investigate the correlations between η_{B0} and the light-neutrino mass matrix $m_\nu = U_\nu m U_\nu^T$.²

Although both of η_{B0} and m_ν seem to depend on the random inputs m_R and m_D in a complicated way, it is not hard to see that there should be no correlation between η_{B0} and U_ν (This was also pointed out in [25]). Recall that we parametrize m_R and m_D as in eq. (2.2)–(2.3). And due to the decomposition eq. (2.4)–(2.5), there are five independent random matrices: U_1, U_2, U_R, D_0 and D_R . The first thing to observe is that changing U_1 with the other four matrices fixed will not affect η_{B0} . This is because:

1. The baryon asymmetry η_{B0} we have been computing in this paper is the total baryon asymmetry, including all the three generations. So m_D enters the calculation of leptogenesis only through the form of the matrix

$$K \equiv h^\dagger h = \left(\frac{\sqrt{2}}{v} \right)^2 U_N^T m_D^\dagger m_D U_N^*,\tag{4.1}$$

with $m_D = \mathcal{D} \cdot U_1 D_0 U_2^\dagger$. Obviously U_1 cancels in K .

2. Throughout the simulation, we are also applying a built-in cut $\Delta m_l^2 = 2.5 \times 10^{-3} eV^2$ by choosing the value of \mathcal{M} to force it. Due to this cut, m_D can potentially affect η_{B0} through the value of \mathcal{M} . However, since the actual relation is

$$\begin{aligned}m_\nu &= m_D m_R^{-1} m_D^T \\ &= \frac{\mathcal{D}^2}{\mathcal{M}} U_1 D_0 U_2^\dagger m_R^{-1} U_2^* D_0 U_1^T \\ &= U_\nu m U_\nu^T,\end{aligned}\tag{4.2}$$

²The correlation between leptogenesis and light-neutrino parameters has been recently studied in [25]. They have some overlap with our results.

we see that changing U_1 would only affect U_ν , not m . So no further adjustment of \mathcal{M} is needed when we change U_1 .

The second point to observe is that any change in U_ν can be achieved by a left translation

$$U_{\nu a} \rightarrow U_{\nu b} = (U_{\nu b} U_{\nu a}^{-1}) U_{\nu a} \equiv U_L U_{\nu a} \quad (4.3)$$

This in turn, can be accounted for by just a left translation in U_1 : $U_{1a} \rightarrow U_{1b} = U_L U_{1a}$, with the other four random matrices unchanged (see eq. (4.2)). This left translation in U_1 is thus a one-to-one mapping between the sub-sample generating $U_{\nu a}$ and the sub-sample generating $U_{\nu b} = U_L U_{\nu a}$. Any two events connected through this one-to-one mapping generate the same value of η_{B0} , because changing U_1 does not change η_{B0} . In addition, the two events have the same chance to appear, because the measure over U_1 is the Haar measure, which is invariant under the left translation. Thus the sub-sample with $U_\nu = U_{\nu a}$ and $U_\nu = U_{\nu b}$, for any arbitrary $U_{\nu a}$ and $U_{\nu b}$, will give the same distribution of η_{B0} , namely that η_{B0} is independent of U_ν . So immediately we conclude that η_{B0} cannot be correlated with the three mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$, the CP phase δ_{CP} , or the phases χ_1, χ_2 .

Three of the Mixing-Split cuts applied to η_{B0} are cuts on mixing angles which we just showed not correlated with η_{B0} . So clearly, the enhancement of η_{B0} is due to its non-zero correlation with R . To study more detail about the correlation between η_{B0} and the light-neutrino masses m , we apply a χ^2 test of independence numerically to the joint distribution between η_{B0} and quantities related to m , including $\lg R$, m_{eff} and m_{total} . For each quantity with η_{B0} , we construct a discrete joint distribution by counting the number of occurrences O_{ij} ($i, j = 1, \dots, 10$) in an appropriate 10×10 partitioning grid. Then we obtain the expected number of occurrences E_{ij} as

$$E_{ij} = \frac{1}{n} \left(\sum_{c=1}^{10} O_{ic} \right) \left(\sum_{r=1}^{10} O_{rj} \right), \quad (4.4)$$

where n is the total number of occurrences in all 10×10 partitions. If the two random variables in question were independent of each other, we would have the test statistic

$$X = \sum_{i,j=1}^{10} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}, \quad (4.5)$$

satisfying the χ^2 distribution with degrees of freedom $(10 - 1) \times (10 - 1) = 81$. We then compute the probability $P(\chi^2 > X)$ for the hypothesis distribution $\chi^2(81)$ to see if the independence hypothesis is likely. Our results from a $n = 3,000,000$ sample Monte Carlo are shown in table 2.

Unambiguously, η_{B0} has nonzero correlations with $\lg R$, m_{eff} , and m_{total} . To see the tendency of the correlations, we draw scatter plots with 5,000 occurrences (figure 8). The plots show that all the three quantities are negatively correlated with η_{B0} . For example the left panel of figure 8 tells us that a smaller $\lg R$ would favor a larger η_{B0} . This explains the slight enhancement of η_{B0} after applying Mixing-Split cuts. But as the scatter plots show, the correlations are rather weak.

	X	$P(\chi^2 > X)$
$\lg R$	1.43×10^5	3.81×10^{-30908}
m_{eff}	8.06×10^3	1.24×10^{-1655}
m_{total}	1.04×10^5	7.24×10^{-22445}

Table 2. χ^2 test of independence between η_{B0} and $\lg R$, m_{eff} , m_{total} .

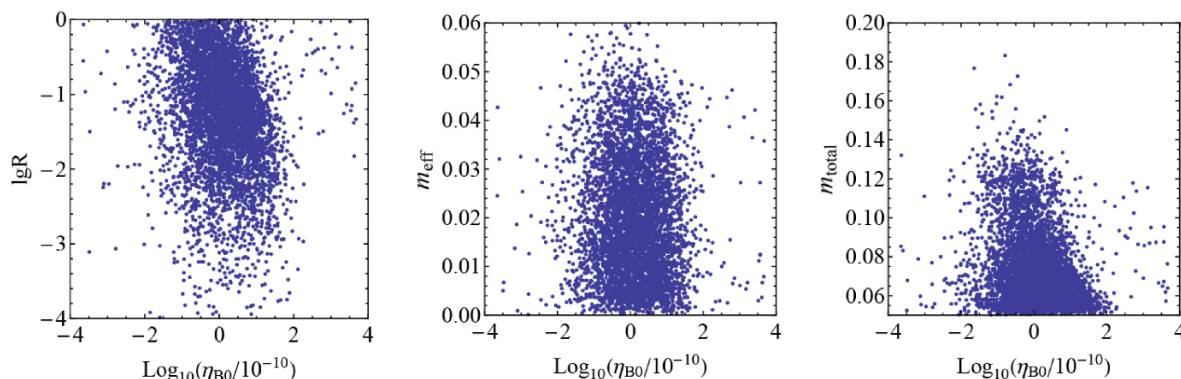


Figure 8. Scatter plots for η_{B0} with $\lg R$, m_{eff} , and m_{total} . Each plot shows a sample of 5,000 occurrences.

5 Possible consequences of a large m_{total}

As mentioned previously, a recent BOSS analysis suggests m_{total} possibly quite large, $m_{\text{total}} = 0.36 \pm 0.10 \text{ eV}$ [26]. Currently their uncertainty is still large, and thus no conclusive argument can be made. If in future the uncertainty pins down near its current central value, anarchy prediction (figure 4) would be obviously inconsistent with it and becomes ruled out. On the other hand, if the central value also comes down significantly, it could still be well consistent with anarchy prediction.

Without the knowledge of future data, we would like to answer the following question: assuming the future data be consistent with anarchy, could a relatively large m_{total} dramatically change anarchy's predictions on other quantities? For this purpose, we introduce a heuristic m_{total} cut:

$$m_{\text{total}} > 0.1 \text{ eV}, \quad (5.1)$$

just to get a sense of how much our predictions could be changed if there turns out to be a large but still consistent m_{total} .

We collect 10^4 occurrences that pass both the Mixing-Split cuts and the m_{total} cut. It turns out that the predictions change quite significantly. We see from figure 2 that the mass hierarchy prediction is overturned, with normal hierarchy only 40% and inverted scenario more likely. This can be expected from figure 6. The predictions of m_{eff} and η_{B0} are shown in figure 9. We see that m_{eff} exhibits a very interesting bipolar distribution. Its overall expectation value also becomes about an order of magnitude larger than before and thus much less challenging to the neutrinoless double beta decay experiments. The prediction

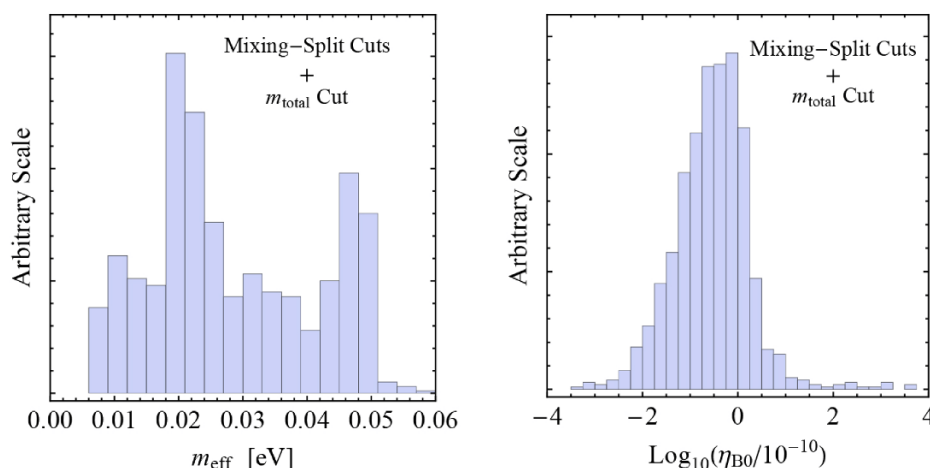


Figure 9. Histograms of m_{eff} (left) and η_{B0} (right) with 10^4 occurrences passed both Mixing-Split cuts and the m_{total} cut.

on η_{B0} drops by about an order of magnitude, but the observed baryon asymmetry is still very likely to be achieved.

6 Conclusions

We have shown that basis independence and free entry independence lead uniquely to Gaussian measure for m_R and m_D . We also showed that an approximate U(1) flavor symmetry can make leptogenesis feasible for neutrino anarchy. Combining the two, we find anarchy model successfully generate the observed amount of baryon asymmetry. Same sampling model is used to study other quantities related to neutrino masses. We found the chance of normal mass hierarchy is as high as 99.9%. The effective mass of neutrinoless double beta decay m_{eff} would probably be well beyond the current experimental sensitivity. The neutrino total mass m_{total} is a little more optimistic. Correlations between baryon asymmetry and light-neutrino quantities were also investigated. We found η_{B0} not correlated with light-neutrino mixings or phases, but weakly correlated with R , m_{eff} , and m_{total} , all with negative correlation. Possible implications of recent BOSS analysis result have been discussed.

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A Basis independence and free entry independence uniquely lead us to Gaussian measure

Let us abstractly write all choices of measure in the form

$$dm = \left(\prod_{ij} dm_{ij} \right) \cdot e^{-f(\{m_{ij}\})}, \quad (\text{A.1})$$

where m stands for m_R or m_D , \prod_{ij} and $\{m_{ij}\}$ run over all the free entries of m . We want the form of $f(\{m_{ij}\})$ so that the measure above has both basis independence and independence among m_{ij} .

Let us first consider m_D . For $N \times N$ m_D , there are N^2 free entries: $m_{11}, m_{12}, \dots, m_{NN}$. For convenience, let us rename them as x_1, x_2, \dots, x_n , where $n = N^2$. Then $\{x_i\}$ forms an irreducible unitary representation of the basis transformation group $U(3)_L \times U(3)_R$ in flavor space:

$$m_D \rightarrow m'_D = U_L m_D U_R^\dagger, \quad (\text{A.2})$$

$$x \rightarrow x' = \Lambda x, \quad (\text{A.3})$$

where $\Lambda = U_L \otimes U_R^*$ is obviously unitary.

Independence of $\{m_{ij}\}$, namely $\{x_i\}$, requires $f(\{m_{ij}\})$ having the form

$$f(\{x_i\}) = f_1(x_1) + f_2(x_2) + \dots + f_n(x_n). \quad (\text{A.4})$$

Here since x_i are complex arguments, all the functions $f_i(x_i)$ are actually abbreviations of $f_i(x_i, x_i^*)$. Under the transformation of eq. (A.3), basis independence requires

$$f_1(x'_1) + \dots + f_n(x'_n) = f_1(x_1) + \dots + f_n(x_n). \quad (\text{A.5})$$

Taking a derivative with respect to x_i^* yields

$$\sum_j \Lambda_{ji}^* \frac{\partial f_j(x'_j)}{\partial x'^{*}_j} = \frac{\partial f_i(x_i)}{\partial x_i^*}. \quad (\text{A.6})$$

Since Λ is unitary, this is the same as

$$\frac{\partial f_i(x_i)}{\partial x'^{*}_i} = \sum_j \Lambda_{ij} \frac{\partial f_j(x_j)}{\partial x_j^*}. \quad (\text{A.7})$$

Thus $\frac{\partial f_i(x_i)}{\partial x_i^*}$ transform in the same way as x_i . Because x_i forms an irreducible representation of the transformation eq. (A.3), the only possibility for $\frac{\partial f_i(x_i)}{\partial x_i^*}$ is

$$\begin{pmatrix} \frac{\partial f_1(x_1)}{\partial x_1^*} \\ \vdots \\ \frac{\partial f_n(x_n)}{\partial x_n^*} \end{pmatrix} = c_a \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad (\text{A.8})$$

with c_a an arbitrary constant. Similarly, taking a derivative of eq. (A.5) with respect to x_i will give us

$$\begin{pmatrix} \frac{\partial f_1(x_1)}{\partial x_1} \\ \vdots \\ \frac{\partial f_n(x_n)}{\partial x_n} \end{pmatrix} = c_b \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix}. \quad (\text{A.9})$$

Combining eq. (A.8) and (A.9) we get

$$\begin{aligned} f(\{m_{D,ij}\}) &= c_1(x_1x_1^* + \cdots + x_nx_n^*) + c_2 \\ &= c_1 \left(\sum_{ij} |m_{D,ij}|^2 \right) + c_2. \end{aligned} \quad (\text{A.10})$$

An important condition in this proof is that x forms an irreducible unitary representation of the basis transformation group, $U(3)_L \times U(3)_R$ in the case of m_D . For the case of m_R , this condition still holds. The relevant basis transformation group for m_R is just $U(3)_R$

$$m_R \rightarrow m'_R = U_R m_R U_R^T, \quad (\text{A.11})$$

$$x \rightarrow x' = \Lambda x. \quad (\text{A.12})$$

$\Lambda = U_R \otimes U_R$ is reducible in general: $3 \otimes 3 = 6 \oplus 3$, but our m_R is symmetric by definition, which only forms the irreducible subspace “6” (Note that if m_R were real symmetric, this symmetric subspace “6” would be further reducible.). So same as in eq. (A.10), we get

$$\begin{aligned} f(\{m_{R,ij}\}) &= c_1(x_1x_1^* + \cdots + x_nx_n^*) + c_2 \\ &= c_1 \left(\sum_i |m_{R,ii}|^2 + 2 \sum_{i < j} |m_{R,ij}|^2 \right) + c_2. \end{aligned} \quad (\text{A.13})$$

In eq. (A.10) and (A.13), c_1 corresponds to the freedom of adjusting \mathcal{D} and \mathcal{M} , while c_2 is just an overall normalization factor. Plugging them back into eq. (A.1), we get the Gaussian measure of m_D and m_R as in eq. (2.8) and (2.9).

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